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On a class of representations of the Yangian and moduli space of monopoles

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Abstract

A new class of infinite dimensional representations of the Yangians $Y(\mathfrak{g})$ and $Y(\mathfrak{b})$ corresponding to a complex semisimple algebra $\mathfrak{g}$ and its Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is constructed. It is based on the generalization of the Drinfeld realization of $Y(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{gl}(N)$ in terms of quantum minors to the case of an arbitrary semisimple Lie algebra $\mathfrak{g}$. The Poisson geometry associated with the constructed representations is described. In particular it is shown that the underlying symplectic leaves are isomorphic to the moduli spaces of G-monopoles defined as the components of the space of based maps of $\mathbb{P}^1$ into the generalized flag manifold $X = G/B$. Thus the constructed representations of the Yangian may be considered as a quantization of the moduli space of the monopoles.

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1 Introduction

The Yangian $Y(\mathfrak{g})$ for a semisimple complex Lie algebra $\mathfrak{g}$ was introduced by Drinfeld as a certain deformation of $U(\mathfrak{g}[t])$ as a Hopf algebra [1], [2], [3] (see for recent review [4], and [5]). Recently a construction of the special class of infinite-dimensional representations of $Y(\mathfrak{gl}(N))$ and $Y(\mathfrak{sl}(N))$ based on the generalization of the Gelfand-Zetlin construction was introduced in [6] (see also [7]). In this paper we generalize this construction to $Y(\mathfrak{g})$ for an arbitrary semisimple Lie algebra $\mathfrak{g}$. This generalization is based on the proposed generalization of the Drinfeld realization of $Y(\mathfrak{gl}(N))$ in terms of quantum minors to the case of $Y(\mathfrak{g})$ for an arbitrary semisimple Lie algebra $\mathfrak{g}$. We also construct representations of $Y(\mathfrak{b})$ where $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra of $\mathfrak{g}$. One should note that not all of the considered representations of $Y(\mathfrak{b})$ may be lifted to the representations of $Y(\mathfrak{g})$. We also describe the Poisson geometry behind the constructed representation by defining explicitly the symplectic leaves of the classical versions of the Yangians $Y(\mathfrak{g})$ and $Y(\mathfrak{b})$. The proposed description of the symplectic leaves reveals a deep connection with the moduli space of G-monopoles, $Lie(G) = \mathfrak{g}$. The symplectic leaves as symplectic manifolds turn out to be open parts of the moduli space of monopoles with maximal symmetry breaking supplied with the symplectic structure introduced in [8] for $G = SU(2)$, in [9] for $G = SU(N)$, and in [10] for an arbitrary semisimple Lie group G. Thus our construction of the Yangian representations can be considered as a quantization of the moduli spaces of G-monopoles.

The results of this paper support the strong connection between quantum integrable systems and problems of the quantization of various moduli spaces. The demonstrated connection between the Yangian and the quantization of the moduli space of the monopoles is a particular example of this deep relation. We plan to consider its implications to the theory of quantum integrable theories elsewhere. Let us remark that the connection between the Atiyah-Hitchin symplectic structure on the moduli space of the $SU(2)$ monopoles [8] with some particular integrable systems was noted previously in [11], [12].

Finally note that the explicit construction of the representations of $Y(\mathfrak{g})$ discussed below appears to be similar to the constructions of the representations of a class of elliptic algebras proposed in [13], [14]. Nevertheless, the main result (Theorem 3.1) seems to be new.

The plan of the paper is as follows. In Section 2 we provide various descriptions of the $Y(\mathfrak{g})$ in terms of the generators and relations. The construction of $Y(\mathfrak{g})$ in terms of the generators $A_i(u), B_i(u), C_i(u)$ for a general Lie algebra is proposed. In Section 3, we describe a particular class of representations of the $Y(\mathfrak{g})$ and $Y(\mathfrak{b})$ and give an explicit realization of its generators in terms of difference operators. The main result is formulated in the Theorem 3.1. In Section 4 we discuss the underlying Poisson geometry and provide a description of the corresponding symplectic leaves of the classical counterpart of the Yangian. It appears that there is an isomorphism between the open part of the symplectic leaves for $Y(\mathfrak{b})$ and the open part of the moduli space of the G-monopoles with the maximal symmetry breaking.

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2 The various presentations of $Y(\mathfrak{g})$

We start with the definition of the Yangian for a semisimple Lie algebra $\mathfrak{g}$ due to Drinfeld [1], [2] (see also [16]) in the form given in [17].

Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ be a simple finite dimensional Lie algebra $\mathfrak{g}$ of rank ℓ over $\mathbb{C}$ with a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$. Let $a = ||a_{ij}||, i, j = 1, \dots, \ell$ be the Cartan matrix of $\mathfrak{g}$, Γ be the set of vertices of the Dynkin diagram of $\mathfrak{g}$, $\{\alpha_i \in \mathfrak{h}^*, i \in \Gamma\}$ be the set of simple roots and $\{\alpha_i^\vee, i \in \Gamma\}$ be the set of the corresponding co-roots ($a_{ij} = \alpha_i^\vee(\alpha_j)$). There exist positive integers $d_1, \dots, d_\ell$ such that the matrix $||d_i a_{ij}||$ is symmetric. Define the invariant bilinear form on $\mathfrak{h}^*$ by $(\alpha_i, \alpha_j) = d_i a_{ij}$, then $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$.

It is convenient to define the generators of the Yangian in terms of the generating series $H_i(u)$, $E_i(u)$, and $F_i(u)$, $i \in \Gamma$:

$$\begin{gathered} H_i(u) = 1 + \sum_{s=0}^{\infty} H_i^{(s)} u^{-s-1} \,, \\ E_i(u) = \sum_{s=0}^{\infty} E_i^{(s)} u^{-s-1} \,, \qquad F_i(u) = \sum_{s=0}^{\infty} F_i^{(s)} u^{-s-1} \,. \end{gathered} \tag{2.1}$$

Definition 2.1 *The Yangian $Y(\mathfrak{g})$ is the associative algebra with generators $H_i^{(s)}, E_i^{(s)}, F_i^{(s)}$, $i \in \Gamma; s = 0, 1, \dots$ and the following defining relations*

$$[H_i(u), H_j(v)] = 0 \,, \tag{2.2}$$

$$\begin{gathered} [H_i(u), E_j(v)] = -\frac{\imath\hbar}{2}\,(\alpha_i, \alpha_j)\,\frac{[H_i(u), E_j(u) - E_j(v)]_+}{u - v} \,, \\ [H_i(u), F_j(v)] = \frac{\imath\hbar}{2}\,(\alpha_i, \alpha_j)\,\frac{[H_i(u), F_j(u) - F_j(v)]_+}{u - v}, \end{gathered} \tag{2.3}$$

$$[E_i(u), F_j(v)] = -\imath\hbar\,\frac{H_i(u) - H_i(v)}{u - v}\,\delta_{i,j} \,, \tag{2.4}$$

$$
\begin{aligned}
[E_i(u),E_i(v)] &= -\frac{\imath\hbar}{2}(\alpha_i,\alpha_i)\frac{(E_i(u)-E_i(v))^2}{u-v}\,,\\
[F_i(u),F_i(v)] &= \frac{\imath\hbar}{2}(\alpha_i,\alpha_i)\frac{(E_i(u)-E_i(v))^2}{u-v}\,,\\
[E_i(u),E_j(v)] &= -\frac{\imath\hbar}{2}(\alpha_i,\alpha_j)\frac{[E_i(u),E_j(u)-E_j(v)]_+}{u-v}-\frac{[E_i^{(0)},E_j(u)-E_j(v)]}{u-v}\,,\\
[F_i(u),F_j(v)] &= \frac{\imath\hbar}{2}(\alpha_i,\alpha_j)\frac{[F_i(u),F_j(u)-F_j(v)]_+}{u-v}-\frac{[F_i^{(0)},F_j(u)-F_j(v)]}{u-v}\,,\\
& i\neq j,\ a_{ij}\neq 0;
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_n}[E_i(u_{\sigma(1)}),[E_i(u_{\sigma(2)}),\ldots,[E_i(u_{\sigma(n)}),E_j(v)]\ldots]]=0\,,\\
&\sum_{\sigma\in\mathfrak{S}_n}[F_i(u_{\sigma(1)}),[F_i(u_{\sigma(2)}),\ldots,[F_i(u_{\sigma(n)}),F_j(v)]\ldots]]=0\,,\\
&\qquad i\neq j,\ n=1-a_{ij}\,,
\end{aligned}
\tag{2.6}
$$

where $[a,b]_+:=ab+ba$.

Denote $Y(\mathfrak{b})\subset Y(\mathfrak{g})$ the subalgebra generated by $H_i(u),E_i(u)$, $i\in\Gamma$.

There is another closely related set of generators of the Yangian which were given by Drinfeld [3] in the case $Y(\mathfrak{sl}(\ell+1))$. Below we propose the generalization of this description to $Y(\mathfrak{g})$ for an arbitrary semisimple Lie algebra $\mathfrak{g}$. In sequel we will use the following convention: $\prod_{s=j}^{k} f_s=1$, for any f_s if $k\leq j$ which help to write down the formulas in a compact way.

Lemma 2.1 *Any generation series $H_i(u)$ of the form (2.1) can be represented in the following form*

$$
H_i(u)=\frac{\prod_{s\neq i}\prod_{r=1}^{-a_{si}}A_s\big(u-\frac{\imath\hbar}{2}(\alpha_i+r\alpha_s,\alpha_s)\big)}{A_i(u)A_i(u-\frac{\imath\hbar}{2}(\alpha_i,\alpha_i))}\,,\quad i=1,\ldots,\ell.
\tag{2.7}
$$

where $A_i(u)$, $i=1,\ldots,\ell$ are formal series $A_i(u)=1+\sum_{s=0}^{\infty}A_i^{(s)}u^{-s-1}$.

Proof: Expanding the left and right hand sides of (2.7) in powers u^{-1} and equating the coefficients in u^{-r-1} one can check that

$$
H_i^{(r)}=\sum_{s=1}^{\ell}a_{si}A_s^{(r)}+f(A_i^{(0)},\cdots,A_i^{(r-1)}).
\tag{2.8}
$$

Since the Cartan matrix is invertible, the statement follows by the induction procedure ■

Let us introduce the generating series $C_i(u)$ and $B_i(u)$:

$$B_i(u) = d_i^{1/2} A_i(u) E_i(u), \quad C_i(u) = d_i^{1/2} F_i(u) A_i(u)\,, \quad i \in \Gamma\,, \tag{2.9}$$

where $d_i = (\alpha_i, \alpha_i)/2$. Due to (2.7), the change of the generators $H_i(u), E_i(u), F_i(u)$ to $A_i(u), C_i(u), B_i(u)$ is invertible. It is convenient to introduce an additional set of series $D_i(u)$. For any $i \in \Gamma$ let

$$D_i(u) = A_i(u) H_i(u) + C_i(u) A_i^{-1}(u) B_i(u)\,, \tag{2.10}$$

where $H_i(u)$ are expressed in terms of $A_i(u)$ in (2.7). Straightforward calculations lead to the following statement.

Proposition 2.1 *For any $i = 1, \ldots, \ell$ the series $A_i(u), B_i(u), C_i(u), D_i(u)$ satisfy the relations*

$$[A_i(u), A_j(v)] = 0\,, \tag{2.11}$$

$$\begin{aligned}
&[A_i(u), B_j(v)] = [A_i(u), C_j(v)] = 0\,, && (i \neq j)\,,\\
&[B_i(u), B_j(v)] = [C_i(u), C_j(v)] = 0\,, && (a_{ij} = 0,\ i \neq j)\,,\\
&[B_i(u), B_i(v)] = [C_i(u), C_i(v)] = 0\,, && \\
&[B_i(u), C_j(v)] = 0\,, && (i \neq j)\,,
\end{aligned} \tag{2.12}$$

$$(u - v)[A_i(u), B_i(v)] = \imath\hbar d_i \big(B_i(u) A_i(v) - B_i(v) A_i(u)\big)\,, \tag{2.13}$$

$$(u - v)[A_i(u), C_i(v)] = \imath\hbar d_i \big(A_i(u) C_i(v) - A_i(v) C_i(u)\big)\,, \tag{2.14}$$

$$(u - v)[B_i(u), C_i(v)] \ = \ \imath\hbar d_i \big(A_i(u) D_i(v) - A_i(v) D_i(u)\big)\,, \tag{2.15}$$

$$(u - v)[B_i(u), D_i(v)] = \imath\hbar d_i \big(B_i(u) D_i(v) - B_i(v) D_i(u)\big)\,, \tag{2.16}$$

$$(u - v)[C_i(u), D_i(v)] = \imath\hbar d_i \big(D_i(u) C_i(v) - D_i(v) C_i(u)\big)\,, \tag{2.17}$$

$$(u - v)[A_i(u), D_i(v)] = \imath\hbar d_i \big(B_i(u) C_i(v) - B_i(v) C_i(u)\big)\,. \tag{2.18}$$

In addition, the following analog of the quantum determinant relation holds:

$$\begin{gathered}
A_i(u) D_i(u - \tfrac{\imath\hbar}{2}(\alpha_i, \alpha_i)) - B_i(u) C_i(u - \tfrac{\imath\hbar}{2}(\alpha_i, \alpha_i)) \ = \\
\prod_{s \neq i} \prod_{r=1}^{-a_{si}} A_s(u - \tfrac{\imath\hbar}{2}(\alpha_i + r\alpha_s, \alpha_s))\,.
\end{gathered} \tag{2.19}$$

The described set of relations between $A_i(u), B_i(u), C_i(u)$ and $D_i(u)$ is a generalization of the $Y(\mathfrak{sl}(\ell + 1))$ relations of [18] to the case of $Y(\mathfrak{g})$ with an arbitrary simple Lie algebra $\mathfrak{g}$.

By (2.11), the coefficients of the generating series $A_i(u)$, $i = 1, \ldots, \ell$ define a commutative subalgebra of $Y(\mathfrak{g})$. This property will be important in the construction of the representations of the Yangian in the next section.

3 Construction of the representations of $Y(\mathfrak{g})$ and $Y(\mathfrak{b})$

In this section the explicit construction of the representations of the Yangian in terms of difference operators is given. The explicit description of the representation of the Yangian in terms of difference/differential operators acting in some functional space is based on the choice of a commutative subalgebra. The elements of this subalgebra act in this representation by multiplication on functions. The proposed construction uses the subalgebra generated by $A_i(u)$ as a distinguished commutative subalgebra. However, we start with explicit description of the resulting representation and then we make some comments how it could be derived starting with the representation of commutative subalgebra generated by $A_i(u)$ and using the commutation relations between $A_i(u), B_i(u)$, and $C_i(u)$ described in Section 2.

Let us introduce a set of variables $\{\gamma_{i,k}\ ; i \in \Gamma;\ k = 1, \ldots, m_i\}$, where m_i are arbitrary positive integer numbers and let $\mathcal{M}$ be the space of meromorphic functions in these variables. Let us define the following difference operators acting on $\mathcal{M}$:

$$\beta_{i,k} = e^{\frac{\imath\hbar}{2}(\alpha_i,\alpha_i)\partial_{\gamma_{i,k}}}. \tag{3.1}$$

where $\partial_{\gamma_{i,k}} := \frac{\partial}{\partial\gamma_{i,k}}$ is a differentiation over $\gamma_{i,k}$. It is useful to arrange the variables into the set of polynomials of the formal variable u of degrees m_i, $i \in \Gamma$

$$P_i(u) = \prod_{p=1}^{m_i}(u - \gamma_{i,p})\,, \qquad i \in \Gamma. \tag{3.2}$$

Consider the operators

$$H_i(u) = R_i(u)\frac{\prod\limits_{s\neq i}\prod\limits_{r=1}^{-a_{si}} P_s\big(u - \frac{\imath\hbar}{2}(\alpha_i + r\alpha_s, \alpha_s)\big)}{P_i(u)P_i(u - \frac{\imath\hbar}{2}(\alpha_i,\alpha_i))}\,, \tag{3.3}$$

$$E_i(u) = d_i^{-1/2}\sum_{k=1}^{m_i}\frac{\prod\limits_{s=i+1}^{\ell}\prod\limits_{r=1}^{-a_{si}} P_s(\gamma_{i,k} - \frac{\imath\hbar}{2}(\alpha_i + r\alpha_s, \alpha_s))}{(u - \gamma_{i,k})\prod\limits_{p\neq k}(\gamma_{i,k} - \gamma_{i,p})}\,\beta_{i,k}^{-1}\,, \tag{3.4}$$

$$F_i(u) = -d_i^{-1/2}\sum_{k=1}^{m_i} R_i(\gamma_{i,k} + \tfrac{\imath\hbar}{2}(\alpha_i,\alpha_i)) \times$$
$$\frac{\prod\limits_{s=1}^{i-1}\prod\limits_{r=1}^{-a_{si}} P_s(\gamma_{i,k} - \frac{\imath\hbar}{2}(\alpha_i + r\alpha_s, \alpha_s) + \frac{\imath\hbar}{2}(\alpha_i,\alpha_i))}{(u - \gamma_{i,k} - \frac{\imath\hbar}{2}(\alpha_i,\alpha_i))\prod\limits_{p\neq k}(\gamma_{i,k} - \gamma_{i,p})}\,\beta_{i,k}\,, \tag{3.5}$$

$$i = 1, \ldots, \ell,$$

where $R_i(u)$ will be specified below.

Theorem 3.1 *(i). For any set of integer numbers* $\{m_i\}$ *satisfying the condition* $l_i := \sum_{j=1}^{\ell} m_j a_{ji} \in \mathbb{Z}_+$, *consider the polynomials*

$$R_i(u) = \prod_{k=1}^{l_i} (u - \nu_{i,k}), \tag{3.6}$$

where $\{\nu_{i,k}\,, i \in \Gamma\,, k = 1, \ldots, l_i\}$ *is a set of arbitrary complex parameters. Then the operators (3.3)-(3.5) considered as formal power series in* u^{-1}, *form a representation of* $Y(\mathfrak{g})$ *in the space* $\mathcal{M}$. *This representation is parameterized by a choice of* $\{m_i\}$ *obeying the above restrictions* ($l_i \geq 0$) *and by arbitrary complex parameters* $\{\nu_{i,k}\}$.

(ii). Let $\{m_i\}$ *be arbitrary integers and* $R_i(u)$ *be rational functions of the following form*

$$R_i(u) = \frac{\prod\limits_{k_+=1}^{l_i^+} (u - \nu^+_{i,k_+})}{\prod\limits_{k_-=1}^{l_i^-} (u - \nu^-_{i,k_-})}, \tag{3.7}$$

where $\{\nu^{\pm}_{i,k_\pm}\,, i \in \Gamma\,, k = 1, \ldots, l_i^{\pm}\}$ *is a set of arbitrary complex parameters and* $l_i^+ - l_i^- = \sum_{j=1}^{\ell} m_j a_{ji}$. *Then the operators (3.3)-(3.4) considered as formal power series in* u^{-1}, *form a representation of* $Y(\mathfrak{b})$ *in the space* $\mathcal{M}$. *This representation is parameterized by the choice of* $\{m_i\}$ *and by arbitrary complex parameters* $\{\nu^{\pm}_{i,k}\}$.

Proof. To prove the theorem introduce the following difference operators

$$\varkappa_{i,k} = d_i^{-1/2}\, \frac{\prod\limits_{s=i+1}^{\ell} \prod\limits_{r=1}^{-a_{si}} P_s(\gamma_{i,k} - \frac{\imath\hbar}{2}(\alpha_i + r\alpha_s, \alpha_s))}{\prod\limits_{p\neq k} (\gamma_{i,k} - \gamma_{i,p})}\, \beta^{-1}_{i,k}\,, \tag{3.8}$$

and

$$\varkappa'_{i,k} = -d_i^{-1/2} R_i(\gamma_{ik} + \tfrac{\imath\hbar}{2}(\alpha_i, \alpha_i)) \frac{\prod\limits_{s=1}^{i-1} \prod\limits_{r=1}^{-a_{si}} P_s(\gamma_{i,k} - \frac{\imath\hbar}{2}(\alpha_i + r\alpha_s, \alpha_s) + \frac{\imath\hbar}{2}(\alpha_i, \alpha_i))}{\prod\limits_{p\neq k} (\gamma_{i,k} - \gamma_{i,p})}\, \beta_{i,k}\,. \tag{3.9}$$

where $R_i(u)$ are rational functions of u. It is easy to show that the operators $\gamma_{i,k}$, $\varkappa_{i,k}$, $\varkappa'_{i,k}$ satisfy the relations

$$\begin{aligned} \varkappa_{i,k}\gamma_{j,l} - \gamma_{j,l}\varkappa_{i,k} &= -\tfrac{\imath\hbar}{2}(\alpha_i, \alpha_i)\delta_{i,j}\delta_{k,l}\varkappa_{i,k}\,, \\ \varkappa'_{i,k}\gamma_{j,l} - \gamma_{jl}\varkappa'_{i,k} &= \tfrac{\imath\hbar}{2}\,(\alpha_i, \alpha_i)\delta_{ij}\delta_{kl}\varkappa'_{i,k}\,, \end{aligned} \tag{3.10}$$

$$\begin{aligned} (\gamma_{i,k} - \gamma_{j,l} - \tfrac{\imath\hbar}{2}(\alpha_i, \alpha_j))\varkappa_{i,k}\varkappa_{j,l} &= (\gamma_{i,k} - \gamma_{j,l} + \tfrac{\imath\hbar}{2}(\alpha_i, \alpha_j))\varkappa_{j,l}\varkappa_{i,k}\,, \\ \varkappa'_{i,k}\varkappa'_{j,l}(\gamma_{i,k} - \gamma_{j,l} + \tfrac{\imath\hbar}{2}\,(\alpha_i, \alpha_j)) &= \varkappa'_{j,l}\varkappa'_{i,k}(\gamma_{i,k} - \gamma_{j,l} - \tfrac{\imath\hbar}{2}\,(\alpha_i, \alpha_j))\,, \end{aligned} \tag{3.11}$$

$$[\varkappa_{i,k},\varkappa'_{j,l}]=0,\;\; i\neq j,\;\; [\varkappa_{i,k},\varkappa'_{i,l}]=0,\;\; k\neq l\,, \tag{3.12}$$

$$\begin{gathered}\varkappa_{i,k}\varkappa'_{i,k}=-d_i^{-1}\frac{R_i(\gamma_{i,k})\prod_{s\neq i}\prod_{r=1}^{-a_{si}}P_s(\gamma_{i,k}-\frac{\imath\hbar}{2}(\alpha_i+r\alpha_s,\alpha_s))}{\prod_{p\neq k}(\gamma_{i,k}-\gamma_{i,p})\prod_{p\neq k}(\gamma_{i,k}-\gamma_{i,p}-\frac{\imath\hbar}{2}(\alpha_i,\alpha_i))}\,,\\ \varkappa'_{i,k}\varkappa_{i,k}=-d_i^{-1}\frac{R_i(\gamma_{i,k}+\frac{\imath\hbar}{2}(\alpha_i,\alpha_i))\prod_{s\neq i}\prod_{r=1}^{-a_{si}}P_s(\gamma_{i,k}-\frac{\imath\hbar}{2}(\alpha_i+r\alpha_s,\alpha_s)+\frac{\imath\hbar}{2}(\alpha_i,\alpha_i))}{\prod_{p\neq k}(\gamma_{i,k}-\gamma_{i,p})\prod_{p\neq k}(\gamma_{i,k}-\gamma_{i,p}+\frac{\imath\hbar}{2}(\alpha_i,\alpha_i))}\,.\end{gathered} \tag{3.13}$$

Now let us define the generators as

$$H_i(u)=R_i(u)\,\frac{\prod_{s\neq i}\prod_{r=1}^{-a_{si}}P_s\big(u-\frac{\imath\hbar}{2}(\alpha_i+r\alpha_s,\alpha_s)\big)}{P_i(u)P_i(u-\frac{\imath\hbar}{2}(\alpha_i,\alpha_i))}\,, \tag{3.14}$$

$$E_i(u)=\sum_{k=1}^{m_i}\frac{1}{u-\gamma_{i,k}}\,\varkappa_{i,k}\,,\quad i\in\Gamma, \tag{3.15}$$

$$F_i(u)=\sum_{k=1}^{m_i}\varkappa'_{i,k}\,\frac{1}{u-\gamma_{i,k}}\,,\quad i\in\Gamma\,, \tag{3.16}$$

and let $R(u_i)$ be rational functions compatible with the expansion (2.1). Then the relations (2.2),(2.3), and (2.5) may be derived by straightforward calculations. If we further restrict $R_i(u)$ to be polynomial functions then the additional relations (2.4) hold. To complete the proof of the theorem one should verify the relations (2.6), which forms in fact the only non-trivial part of the proof. One can see that such a verification is reduced to the following combinatorial lemma. ■

Lemma 3.1 *Let $\gamma_{ik},\varkappa_{ik}$, and $\varkappa'_{ik}$ satisfy the relations (3.10), (3.11). Then for any $n=2,3,4$ the following formulas holds:*

$$\begin{gathered}\sum_{\sigma\in\mathfrak{S}_n}[\varkappa_{i,k_{\sigma(1)}},[\ldots,[\varkappa_{i,k_{\sigma(n)}},\varkappa_{j,l}]]...]\;=\\ =\Big(\frac{\imath\hbar(\alpha_i,\alpha_i)}{2}\Big)^n\prod_{s=0}^{n-1}(a_{ij}+s)\cdot\Big(\sum_{\sigma\in\mathfrak{S}_n}\tilde{\varkappa}_{i,k_{\sigma(1)}}\cdot\ldots\cdot\tilde{\varkappa}_{i,k_{\sigma(n)}}\Big)\cdot\varkappa_{j,l}\,,\end{gathered} \tag{3.17}$$

$$\begin{gathered}\sum_{\sigma\in\mathfrak{S}_n}[\varkappa'_{i,k_{\sigma(1)}},[\ldots,[\varkappa'_{i,k_{\sigma(n)}},\varkappa'_{j,l}]]...]\;=\\ =\Big(\frac{\imath\hbar(\alpha_i,\alpha_i)}{2}\Big)^n\prod_{s=0}^{n-1}(a_{ij}+s)\cdot\Big(\sum_{\sigma\in\mathfrak{S}_n}\tilde{\varkappa}'_{i,k_{\sigma(1)}}\cdot\ldots\cdot\tilde{\varkappa}'_{i,k_{\sigma(n)}}\Big)\cdot\varkappa'_{j,l}\,,\end{gathered} \tag{3.18}$$

where

$$\begin{aligned} \tilde{\varkappa}_{i,k} &:= \frac{1}{\gamma_{i,k} - \gamma_{j,l} + \frac{\imath\hbar(\alpha_i,\alpha_j)}{2}}\, \varkappa_{i,k}\,, \\ \tilde{\varkappa}'_{i,k} &:= \varkappa'_{i,k}\, \frac{1}{\gamma_{i,k} - \gamma_{j,l} + \frac{\imath\hbar(\alpha_i,\alpha_j)}{2}}\ . \end{aligned} \tag{3.19}$$

Proof. We outline the proof of the only non-trivial relations (3.17). Thus, we should calculate the following expression for $n = 2, 3, 4$

$$X_n = \sum_{\sigma\in\mathfrak{S}_n} [\varkappa_{i,k_{\sigma(1)}}, [\ldots, [\varkappa_{i,k_{\sigma(n)}}, \varkappa_{j,l}]]\ldots] \tag{3.20}$$

We consider only the case of non-coincident indexes $k_i \neq k_j$ for any $i \neq j$. The proof of the general case is quite similar. Let $\eta_{ij} = \frac{\imath\hbar(\alpha_i,\alpha_j)}{2}$. For $n = 2, 3, 4$, X_n may be represented as a sum of $n!$ terms of the first type:

$$\eta_{ij}^n \prod_{k=1}^{n} (\gamma_{i,k} - \gamma_{j,m})^{-1} [\varkappa_{i,k_{\sigma(1)}}, [\ldots, [\varkappa_{i,k_{\sigma(n)}}, \varkappa_{j,l}]_+]_+ \ldots]_+ \tag{3.21}$$

and $n^n - n!$ terms of the second type:

$$\begin{gathered} \eta_{ii}^s \eta_{ij}^{n-s} \prod_{k=1}^{s} (\gamma_{i,k} - \gamma_{j,m})^{-1} \prod_{\alpha,\beta} (\gamma_{i,\alpha} - \gamma_{i,\beta})^{-1} \times \\ [\varkappa_{i,k_{\sigma(1)}}, [\ldots, \varkappa_{j,l} [\varkappa_{i,k_{\sigma(\alpha)}}, \varkappa_{i,k_{\sigma(\beta)}}]]_+]_+ \ldots]_+\ . \end{gathered} \tag{3.22}$$

One can reduce the terms of the second type to $\binom{n}{2}$ terms of the first type as follows. Given $I_n := \{1, \ldots, n\}$ and the set of variables $\{a_i,\ i \in I\}$, d the following iterative formula holds.

$$\prod_{i\in I_n} (a_i - d)^{-1} = \sum_{r\in I_n} (a_r - d)^{-1} \prod_{i\in I_n\setminus\{r\}} (a_i - a_r)^{-1}.$$

The left hand side $\prod\limits_{s=1}^{n} (a_s - d)^{-1}$ of this formula is exactly of the first type for $a_s := \gamma_{i,k_s}$ and $d := \gamma_{j,l}$. The iterations on the right hand side coincide with all the terms of the second type (3.22) and thus we are left with only the terms of the first type. The simple transformations then lead to (3.17). ■

Finally, let us briefly explain how these representations naturally arise from the relations (2.11)-(2.19). Due to (2.11) $A_i^{(s)}, i \in \Gamma, s = 0, 1, 2, \ldots$ generate a commutative subalgebra of $Y(\mathfrak{g})$. We would like to construct the representation in the space of functions of the finite collection of the variables $\{\gamma_{ik}\}$ such that $A_i^{(s)}$ act through the multiplication by certain functions of $\{\gamma_{ik}\}$. It is natural to look for the representation of $A_i(u)$ in the form $A_i(u) = X_i(u)P_i(u)$ where $P_i(u)$ are given by (3.2) and $X_i(u) = 1 + \sum_{s=0}^{\infty} X_i^{(s)} u^{-1-s}$ are some γ_{ik}-independent series. From the commutation relations (2.11)-(2.19) one derives that $B_i(\gamma_{i,k})$ and $(C_i(\gamma_{i,k}))^{-1}$ are proportional to the shift operator (3.1). Therefore, by (2.9) the residues of $E_i(u)$ and $F_i(u)$ are proportional to the $B_i(\gamma_{i,k})$ and $C_i(\gamma_{i,k})$ respectively. This explains the ansatz (3.15), and (3.16) for the generators $E_i(u)$ and $F_i(u)$.

4 Symplectic leaves of the Yangian and the monopole moduli spaces

In this section we describe the Poisson geometry relevant to the description of the Yangian representations proposed above. It appears that this leads to the direct connection with moduli spaces of G-monopoles such that $Lie(G) = \mathfrak{g}$.

Let $Y_{cl}(\mathfrak{g})$ and $Y_{cl}(\mathfrak{b})$ be the Poisson algebras corresponding to the classical limit of $Y(\mathfrak{g})$ and $Y(\mathfrak{b})$ in the sense of [20], [21]. The elements $Y_{cl}(\mathfrak{g})$ may be described as functions on the formal loop group LG_{-} based at the trivial loop $g(u) = e$ where $e \in G$ is unity element. We use its parameterization in terms of the infinite series of the form $F(u) = \sum_{s=0}^{\infty} F^{(s)} u^{-s-1}$.

The description of the Poisson algebra $Y_{cl}(\mathfrak{g})$ in terms of the generators and relations can be obtained from (2.2)-(2.6) by taking the limit $\hbar \to 0$:

$$\{h_i(u), h_j(v)\} = 0\,, \tag{4.1}$$

$$\begin{aligned} \{h_i(u), e_j(v)\} &= -\,(\alpha_i, \alpha_j)\,\frac{h_i(u)(e_j(u) - e_j(v))}{u-v}\,,\\ \{h_i(u), f_j(v)\} &= (\alpha_i, \alpha_j)\,\frac{h_i(u)(f_j(u) - f_j(v))}{u-v}\,, \end{aligned} \tag{4.2}$$

$$\{e_i(u), f_j(v)\} = -\,\delta_{ij}\frac{h_i(u) - h_i(v)}{u-v}\,, \tag{4.3}$$

$$\begin{gathered} \{e_i(u), e_i(v)\} = -(\alpha_i, \alpha_i)\frac{(e_i(u) - e_i(v))^2}{u-v},\\ \{f_i(u), f_i(v)\} = (\alpha_i, \alpha_i)\frac{(e_i(u) - e_i(v))^2}{u-v},\\ \{e_i(u), e_j(v)\} = -(\alpha_i, \alpha_j)\frac{e_i(u)(e_j(u) - e_j(v))}{u-v} - \frac{\{e_i^{(0)}, (e_j(u) - e_j(v))\}}{u-v},\\ \{f_i(u), f_j(v)\} = (\alpha_i, \alpha_j)\frac{f_i(u)(f_j(u) - f_j(v))}{u-v} - \frac{\{f_i^{(0)}, (f_j(u) - f_j(v))\}}{u-v},\\ i \neq j, \quad a_{ij} \neq 0; \end{gathered} \tag{4.4}$$

$$\begin{gathered} \sum_{\sigma \in \mathfrak{S}_n} \{e_i(u_{\sigma(1)}), \{e_i(u_{\sigma(2)}), \ldots, \{e_i(u_{\sigma(n)}), e_j(v)\} \ldots\}\} = 0\,,\\ \sum_{\sigma \in \mathfrak{S}_n} \{f_i(u_{\sigma(1)}), \{f_i(u_{\sigma(2)}), \ldots, \{f_i(u_{\sigma(n)}), f_j(v)\} \ldots\}\} = 0\,,\\ n = 1 - a_{ij}, \quad (i \neq j)\,. \end{gathered} \tag{4.5}$$

There exists the following simple interpretation of the generators $h_i(u), e_i(u), f_i(u)$. Let us fix the Gauss decomposition of an element $g(u) \in LG_-$:

$$g(u) = \exp\big(\sum_{\alpha} f_\alpha(u) F_\alpha\big) \cdot \exp\big(\sum_{i=1}^{\ell} \phi_i(u) H_i\big) \cdot \exp\big(\sum_{\alpha} e_\alpha(u) E_\alpha\big)\,, \tag{4.6}$$

where H_i, F_α, E_α provide a basis of $\mathfrak{g}$ labeled by positive roots α, and $\phi_i(u), f_\alpha(u), e_\alpha(u)$ are the local exponential coordinates on the group LG_-. Note that we consider the functions on the formal loop group LG_- (i.e. we deal with functions on the formal neighbourhood of $e \in LG_-$) and thus the Gauss decomposition (4.6) is valid "everywhere". The functions $e_i(u) := e_{\alpha_i}(u)$, $f_i(u) := f_{\alpha_i}(u)$ corresponding to the simple roots α_i together with $h_i(u) = \exp\{-\sum_{j=1}^{\ell} a_{ji}\phi_j(u)\}$ give us a set of the generators satisfying the relations (4.1)-(4.5). The local coordinates $\phi_i(u), e_i(u), f_i(u)$ may be expressed explicitly in terms of the matrix elements of the fundamental representations of $U(\mathfrak{g})$. Let $\{\pi_i\}$ be a set of fundamental representations corresponding to the fundamental weights $\{\omega_i\}$ of $\mathfrak{g}$ and $v^{(i)}_{+/-}$ be the highest/lowest vectors in these representation. Denote by $a_i(u)$, $b_i(u)$, $c_i(u)$, $d_i(u)$ the following formal series

$$\begin{aligned} a_i(u) &= \langle v_-^{(i)} | \pi_i(g(u)) | v_+^{(i)} \rangle\,, \\ b_i(u) &= \langle v_-^{(i)} | \pi_i(g(u)) \pi_i(F_i) | v_+^{(i)} \rangle\,, \\ c_i(u) &= \langle v_-^{(i)} | \pi_i(E_i) \pi_i(g(u)) | v_+^{(i)} \rangle\,, \\ d_i(u) &= \langle v_-^{(i)} | \pi_i(E_i) \pi_i(g(u)) \pi_i(F_i) | v_+^{(i)} \rangle\,. \end{aligned} \tag{4.7}$$

Lemma 4.1 *The coordinates corresponding to the simple roots and Cartan elements entering the Gauss decomposition (4.6) can be expressed through the matrix elements (4.7) as follows*

$$\begin{aligned} e^{\phi_i(u)} &= a_i(u)\,, \\ e_i(u) &= \frac{b_i(u)}{a_i(u)}\,, \\ f_i(u) &= \frac{c_i(u)}{a_i(u)}\,. \end{aligned} \tag{4.8}$$

The variables $a_i(u), b_i(u), c_i(u), d_i(u)$ are the classical counterparts of the variables $A_i(u)$, $B_i(u)$, $C_i(u)$, $D_i(u)$ defined in (2.7),(2.9),(2.10). A similar description of $Y(\mathfrak{gl}(\ell+1))$ was given in [3] (see also [19], [22]). In this case the functions $a_i(u), b_i(u), c_i(u), d_i(u)$ are given by the minors of the matrix $\pi_*(g(u))$ where π_* is the tautological representation $\pi_* : \mathfrak{gl}(\ell + 1) \to \mathrm{End}(\mathbb{C}^{\ell+1})$, which agrees with (4.7). Let us remark that (4.7) may be considered as a classical counterpart of the universal R-matrix map of the dual Hopf algebra with the opposite comultiplication A^0 to the Hopf algebra A (see [2] for details). Together with the explicit Gauss form of the universal R-matrix this should provide the quantum version of Lemma 4.1. The simplest example of $Y(\mathfrak{sl}(2))$ may be extracted from [17].

Now consider the classical counterpart of the Yangian representations constructed in the Section 3. These representations have the following property: the images of $E_i(u)$ are rational operator-valued functions in u with simple poles. Given the Poisson brackets (4.1)-(4.5) on LG_-, the representations of the Yangian correspond to the symplectic leaves in LG_-. A similar description holds for $Y_{cl}(\mathfrak{b})$ in terms of the symplectic leaves in LB_-. We define the symplectic leaf $\mathcal{O}$ to be rational if the restriction of the generators $e_i(u)$ is a rational function over u and let $\mathcal{O}^{(0)} \subset \mathcal{O}$ be an open part corresponding to $e_i(u)$ having only simple poles. Thus the symplectic leaves corresponding to the representations constructed in Section 3 are rational. One can describe the symplectic leaves as symplectic manifolds as follows. Open parts $\mathcal{O}^{(0)}$ of the rational symplectic leaves in LB_- corresponding to the representations constructed in Theorem 3.1 are isomorphic (as abstract manifolds) to the open subsets of the space of $\widetilde{\mathcal{M}}_b(\mathbf{m})$ of the based rational maps

$$e = (e_1, \cdots, e_\ell) : (\mathbb{P}^1, \infty) \xrightarrow{e} (\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{\ell}, 0 \times \cdots \times 0), \tag{4.9}$$

of the fixed multi-degree $\mathbf{m} = (m_1, \cdots, m_\ell)$ where $e_i(u)$ are the generators of $Y_{cl}(\mathfrak{b})$ corresponding to simple roots. Analogously open parts $\mathcal{O}^{(0)}$ of the rational symplectic leaves of LG_- corresponding to the representations constructed in Theorem 3.1 are isomorphic to the open subsets of $\widetilde{\mathcal{M}}_b(\mathbf{m})$ with additional restrictions $\sum_{j=1}^{\ell} m_j a_{ji} = l_i \in \mathbb{Z}_+$.

Taking into account the results of [15] one can reformulate the description of the symplectic leaves as follows. Consider the space $\mathcal{M}(\mathbf{m})$ of the holomorphic maps $\mathbb{P}^1 \to G/B$ of multi-degree $\mathbf{m} = (m_1, \ldots, m_\ell) \in \Lambda_W^\vee$, where $\Lambda_W^\vee = H_2(G/B, \mathbb{Z})$ is the co-weight lattice of $\mathfrak{g}$. It will be useful to consider G/B as a manifold parameterizing the Borel subgroups in G. Choose some Borel subgroup B_+ and let $b_+ \subset G/B$ be the corresponding point in the flag manifold. Let us fix the local coordinate on $\mathbb{P}^1$ and consider the evaluation map $ev_\infty : \mathcal{M}(\mathbf{m}) \longrightarrow G/B$ defined as $ev_\infty : f \to f(\infty)$. Thus $\mathcal{M}(\mathbf{m})$ is supplied with the structure of the fibred space over G/B and the fibre is naturally identified with the moduli space $\mathcal{M}_b(\mathbf{m})$ of the based holomorphic maps $f : (\mathbb{P}^1, \infty) \to (G/B, b_+)$ of the multi-degree $\mathbf{m}$. It appears that the open part of $\mathcal{M}_b(\mathbf{m})$ can be naturally identified with the moduli space $\widetilde{\mathcal{M}}_b(\mathbf{m})$ introduced above. Actually this follows from the results of Drinfeld in the form presented in [15]. Thus it was shown in [15] that $\mathcal{M}_b(\mathbf{m})$ is a smooth manifold of dimension

$$\dim \mathcal{M}_b(\mathbf{m}) = 2|\mathbf{m}| = 2(m_1 + \ldots + m_\ell).$$

The explicit description of the manifold $\mathcal{M}_b(\mathbf{m})$ can be obtained by generalizing the classical Plücker embedding of G/B into the product $\prod_{i \subset \Gamma} \mathbb{P}(V_{\omega_i})$ of the projectivisations of the fundamental representations V_{ω_i} as follows. Let $\pi_\lambda : U(\mathfrak{g}) \to End(V_\lambda)$ be the irreducible representation of the universal enveloping algebra $U(\mathfrak{g})$ with the highest weights λ and $\mathcal{V}_\lambda = V_\lambda \otimes \mathcal{O}_{\mathbb{P}^1}$ be the corresponding trivial vector bundles on $\mathbb{P}^1$. Using the local coordinate we identify $\Gamma(\mathbb{A}^1, \mathcal{V}_\lambda) = V_\lambda \otimes \mathbb{C}[u]$. Denote by v_+^λ the highest weight vectors in V_λ with respect to the Borel subgroup B_+. Similarly let v_-^λ be the lowest weight vector in the dual representation $V_\lambda^\vee$ normalized by the condition $\langle v_-^\lambda | v_+^\lambda \rangle = 1$. We also introduce the additional set of vectors $\pi_\lambda(F_i)v_+^\lambda$, $\pi_\lambda(E_i)v_-^\lambda$ where F_i and E_i are the generators corresponding to the simple roots α_i. Thus given a section $v^\lambda \in \Gamma(\mathbb{A}^1, \mathcal{V}_\lambda)$ we have the following decomposition

$v^\lambda(u) = a_\lambda(u) \cdot v_+^\lambda + \sum_{i\in\Gamma} b_\lambda^i(u)\pi_\lambda(F_i)v_+^\lambda + \phi^\lambda(u)$ with $\phi^\lambda \subset \Gamma(\mathbb{A}^1, \mathcal{V}_\lambda)$ satisfying $\langle v_-^\lambda|\phi^\lambda\rangle = \langle v_-^\lambda|\pi_\lambda(E_i)|\phi^\lambda\rangle = 0$.

According to Drinfeld (see [15] for the details) the moduli space $\mathcal{M}_b(\mathbf{m})$ is isomorphic to the space $Z_{\mathbf{m}}$ of the collections of sections $v^\lambda(u)$ for each $\lambda \in \Lambda_W^+$ satisfying the conditions

1. The polynomial $a_\lambda(u)$ is monic of degree $\langle m, \lambda\rangle$;
2. The degree of $(v^\lambda - a_\lambda(u)v_+^\lambda)$ is strictly less then $\langle m, \lambda\rangle$;
3. For any G-equivariant morphism $\phi : V_\lambda \otimes V_\mu \to V_\nu$ such that $\nu = \mu + \lambda$ and the conjugated morphism satisfies $\phi^*(v_-^\nu) = v_-^\lambda \otimes v_-^\mu$ we have $\phi(v^\lambda \otimes v^\mu) = v^\nu$;
4. For any G-equivariant morphism $\phi : V_\lambda \otimes V_\mu \to V_\nu$ such that $\nu < \mu + \lambda$ we have $\phi(v^\lambda \otimes v^\mu) = 0$.

It is easy to see that the set $\{v^\lambda, \lambda \in \Lambda_W^+\}$ satisfying these conditions is determined by its subset $\{v^{\omega_i}\}$ corresponding to fundamental representations ω_i. Moreover, given arbitrary polynomials $a_{\omega_i}(u)$ and $b_{\omega_i}^i(u)$ satisfying the conditions (1) and (2) above (i.e. $a_{\omega_i}(u)$ are monic and $deg(a_{\omega_i}) = deg(b_{\omega_i}^i) + 1 = m_i$) there exist such $\phi^{\omega_i}(u)$ that for $v^{\omega_i}(u) = a_{\omega_i}(u) \cdot v_+^{\omega_i} + b_{\omega_i}^i(u)\pi_{\omega_i}(F_i)v_+^{\omega_i} + \phi^{\omega_i}(u)$ the conditions (3) and (4) hold (see [15], [10] for details). Let us consider the subset of the polynomials a_{ω_i} and b_{ω_i} such that the roots $\gamma_{i,k}$ of $a_{\omega_i}(u)$ do not coincide $\gamma_{i,k} \neq \gamma_{j,l}$ for $(i,k) \neq (j,l)$. The space of such polynomials $a_i(u) \equiv a_{\omega_i}(u)$, $b_i(u) \equiv b_{\omega_i}^i(u)$ is $2|\mathbf{m}|$-dimensional and thus is isomorphic to the open subspace in the moduli space $\mathcal{M}_b(\mathbf{m})$. Note that the polynomials $a_i(u)$ and $b_i(u)$ define the map $e \in \widetilde{\mathcal{M}}_b(\mathbf{m})$

$$(\mathbb{P}^1, \infty) \xrightarrow{e} (\underbrace{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}_{\ell}, 0 \times \ldots \times 0),$$

given by

$$e(u) = (b_1(u)/a_1(u)) \times \ldots \times (b_\ell(u)/a_\ell(u)).$$

Therefore we have established the isomorphism of the open parts of the moduli spaces

$$\phi : \mathcal{M}_b(\mathbf{m}) \longrightarrow \widetilde{\mathcal{M}}_b(\mathbf{m}).$$

One can summarize this in the following

Proposition 4.1 *(i) The open parts $\mathcal{O}^{(0)}$ of the rational symplectic leaves of $Y_{cl}(\mathfrak{b})$ corresponding to the representations constructed in Theorem 3.1 are isomorphic to the open parts of the spaces of the based maps*

$$(\mathbb{P}^1, \infty) \to (G/B, b_+) \tag{4.10}$$

of the fixed multi-degree $\mathbf{m} = (m_1, \cdots, m_\ell) \in H_2(G/B, \mathbb{Z})$.

(ii) The open parts $\mathcal{O}^{(0)}$ of the rational symplectic leaves of $Y_{cl}(\mathfrak{g})$ corresponding to the representations constructed in Theorem 3.1 are isomorphic to the spaces of the based maps (4.10) with additional restrictions $\sum_{j=1}^{\ell} m_j a_{ji} = l_i \in \mathbb{Z}_+$.

The connection with the explicit parameterization used in the previous sections is as follows. Let us parameterize the open subset $U \subset \widetilde{\mathcal{M}}_b(\mathbf{m})$ by the following étale coordinates

$$(x_{i,k},\, y_{i,k}), \quad i = 1, \dots, \ell, \quad k = 1, \dots, m_i$$

defined by the conditions

$$a_i(x_{i,k}) = 0, \quad y_{i,k} = b_i(x_{i,k}).$$

Then the coordinates (x_{ik}, y_{ik}) are related to the coordinates $(\gamma_{i,k}, \varkappa_{i,k})$ by the simple redefinition

$$x_{i,k} = \gamma_{i,k}\,, \tag{4.11}$$

$$y_{i,k} = \varkappa_{i,k} \prod_{s \neq k} (\gamma_{i,k} - \gamma_{i,s})\,. \tag{4.12}$$

Note that the classical limit of the relations (3.10),(3.11) provides the open part of the space $\mathcal{M}_b(m)$ with the holomorphic symplectic structure

$$\{\gamma_{i,k}, \gamma_{j,l}\} = 0, \tag{4.13}$$

$$\{\gamma_{i,k}, \varkappa_{j,l}\} = \tfrac{1}{2}(\alpha_i, \alpha_i)\delta_{i,j}\delta_{k,l}\varkappa_{j,l}\,, \tag{4.14}$$

$$\{\varkappa_{i,k}, \varkappa_{j,l}\} = (\alpha_i, \alpha_j)\frac{\varkappa_{i,k}\varkappa_{j,l}}{\gamma_{i,k} - \gamma_{j,l}}\,, \quad (i,k) \neq (j,l), \tag{4.15}$$

or equivalently in the coordinates $(x_{i,k}, y_{i,k})$

$$\{x_{i,k}, x_{j,l}\} = 0, \tag{4.16}$$

$$\{y_{i,k}, y_{i,l}\} = 0, \tag{4.17}$$

$$\{x_{i,k}, y_{j,l}\} = \tfrac{1}{2}(\alpha_i, \alpha_i)\delta_{i,j}\delta_{k,l}y_{j,l}, \tag{4.18}$$

$$\{y_{i,k}, y_{j,l}\} = (\alpha_i, \alpha_j)\frac{y_{i,k}y_{j,l}}{x_{i,k} - x_{j,l}}\,, \quad i \neq j. \tag{4.19}$$

In general the symplectic leaves of the Poisson-Lie group G are connected components of the intersections of the double cosets of the Poisson-Lie dual G^* in $G \times G$ with the diagonal $G \subset G \times G$ [21]. We are going to discuss the connection of this description with the algebro-geometric description considered above in the separate publication.

It appears that description of the symplectic leaves of the Poisson-Lie groups associated with the Yangian given in this section provides a direct connection with the moduli spaces of the G-monopoles with the maximal symmetry breaking. These moduli spaces are also given by the spaces of the based maps $\mathbb{P}^1 \to G/B$ [23], [24], [25]. Our construction provides the holomorphic symplectic structure on these spaces. The explicit description of the holomorphic symplectic structure on the moduli space of the monopoles was given in the case of $G = SU(N)$ in [9] (generalizing the results for $SU(2)$ of [8]) and for general case in [10]. It turns out that our description of the coordinates on the moduli space and the expression for the Poisson structure in coordinates $(x_{i,k}, y_{i,k})$ exactly matches the description given in [10]. Thus the representation constructed in Section 2 can be considered as a quantization of the moduli space of the monopoles.

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